Int. J. Solids Structures, 1969, Vol. 5, pp. 281 to 288. Pergamon Press. Printed in Great Britain

# A GENERAL THEORY FOR THE BRANCHING ANALYSIS OF DISCRETE STRUCTURAL SYSTEMS

J. M. T. THOMPSON and A. C. WALKER

Department of Civil Engineering, University College, London

**Abstract**—A general perturbation theory for the branching analysis of discrete conservative structural systems is presented. Such systems are best analysed without resort to a scheme of diagonalization, and the absence of such a scheme distinguishes the present development from earlier studies. The tensor notation and the system of sliding axes employed in earlier studies are however of considerable analytical value and are therefore retained. The theory is presented for both a general and a specialized class of system and some general features of the perturbation scheme are established. The application of the general theory to two branching problems is finally outlined and its merits discussed.

## 1. INTRODUCTION

THE continuum studies of Koiter, Budiansky and Hutchinson and others [1–19] have amply demonstrated the value of branching analyses of initial post-buckling, and it is a corresponding general theory for the branching analysis of *discrete* structural systems—generated for example by the finite-element Rayleigh–Ritz procedure— that is presented here. Such discrete systems are best analysed without resort to a scheme of diagonalization, and it is the absence of such a scheme that distinguishes the present development from the earlier study of Thompson [20] which was primarily designed to illustrate the various branching phenomena. The tensor notation and the system of sliding axes employed in this earlier study are however of considerable analytical value and have been retained.

The application of the general theory to two branching problems is finally outlined and its merits discussed, the two analyses being more fully described in two companion papers [21, 22].

## 2. THEORY FOR THE GENERAL SYSTEM

Following earlier studies [20, 23–27], we consider a conservative structural system described by the total potential energy function $V(Q_i, \Lambda)$, where $Q_i$ represents a set of $n$ generalized coordinates and $\Lambda$ is a loading parameter.

We suppose that in the region of interest the $n$ equilibrium equations $V_i = 0$ yield a single-valued *fundamental* solution $Q_i = Q_i^F(\Lambda)$, a subscript on $V$ denoting partial differentiation with respect to the corresponding generalized coordinate. A *sliding* set of incremental coordinates $q_i$ is then defined by the $n$ equations

$$Q_i = Q_i^F(\Lambda)+q_i, \tag{1}$$

and we introduce the new energy function [20]

$$W[q_i, \Lambda] \equiv V[Q_i^F(\Lambda)+q_i, \Lambda]. \tag{2}$$

The normal equilibrium and stability conditions hold good for this transformed energy function which has the properties

$$\left.\begin{aligned} W_i[0, \Lambda] &= 0, \\ W_i'[0, \Lambda] &= 0, \\ W_i''[0, \Lambda] &= 0, \\ &\dots\dots\dots\dots\dots \\ &\text{etc.,} \end{aligned}\right\} \tag{3}$$

a subscript again denoting partial differentiation with respect to the corresponding generalized coordinate and a prime denoting partial differentiation with respect to $\Lambda$. We see that we have a valid mapping from the original $\Lambda - Q_i$ space to the new $\Lambda - q_i$ space in which the fundamental equilibrium path is given by $q_i = 0$.

Supposing now that a discrete critical point $C$ lies on the fundamental equilibrium path at $\Lambda = \Lambda^C$, the determinant of $W_{ij}$ will vanish at this point so we can write

$$|W_{ij}(0, \Lambda^C)| = 0. \tag{4}$$

We now seek to express any *post-buckling* equilibrium path emerging from this critical equilibrium state in the parametric form [20, 24]

$$q_j = q_j(q_1), \qquad \Lambda = \Lambda(q_1), \tag{5}$$

assuming without any essential loss of generality that the *first* generalized coordinate is a suitable expansion parameter. Here as elsewhere in the paper it is convenient to write $q_1$ as $q_1(q_1)$.

These parametric equations can be substituted into the equilibrium equations $W_i = 0$ to give the identity

$$W_i[q_j(q_1), \Lambda(q_1)] = 0. \tag{6}$$

Here the left-hand side is simply a function of the independent variable $q_1$, so we can differentiate the equations with respect to $q_1$ as many times as we please. Thus differentiating once, twice, and three times we can generate the ordered equilibrium equations

$$W_{ij}q_{j1} + W_i'\Lambda_1 = 0, \tag{7}$$

$$(W_{ijk}q_{k1} + W_{ij}'\Lambda_1)q_{j1} + W_{ij}q_{j11} + (W_{ij}'q_{j1} + W_i''\Lambda_1)\Lambda_1 + W_i'\Lambda_{11} = 0. \tag{8}$$

$$\begin{aligned} &\{(W_{ijkl}q_{l1} + W_{ijk}'\Lambda_1)q_{k1} + W_{ijk}q_{k11} + (W_{ijk}'q_{k1} + W_{ij}''\Lambda_1)\Lambda_1 + W_{ij}'\Lambda_{11}\}q_{j1} \\ &+ (W_{ijk}q_{k1} + W_{ij}'\Lambda_1)q_{j11} + (W_{ijk}q_{k1} + W_{ij}'\Lambda_1)q_{j11} + W_{ij}q_{j111} \\ &+ \{(W_{ijk}'q_{k1} + W_{ij}''\Lambda_1)q_{j1} + W_{ij}'q_{j11} + (W_{ij}''q_{j1} + W_i'''\Lambda_1)\Lambda_1 + W_i''\Lambda_{11}\}\Lambda_1 \\ &+ (W_{ij}'q_{j1} + W_i''\Lambda_1)\Lambda_{11} + (W_{ij}'q_{j1} + W_i''\Lambda_1)\Lambda_{11} + W_i'\Lambda_{111} = 0. \end{aligned} \tag{9}$$

Here a subscript one denotes differentiation with respect to $q_1$, and following the earlier scheme we must remember that

$$\left.\begin{aligned} q_{11} &\equiv \frac{\mathrm{d}q_1}{\mathrm{d}q_1} = 1, \\ q_{111} &\equiv \frac{\mathrm{d}^2 q_1}{\mathrm{d}q_1^2} = 0, \quad \text{etc.} \end{aligned}\right\} \tag{10}$$

The dummy-suffix summation convention is employed with all summations ranging from 1 to $n$.

Evaluating the ordered equilibrium equations at the critical point $q_i = 0$, $\Lambda = \Lambda^C$, and remembering that $W_i''|^C = W_i''|^C = 0$ we have

$$\left.\begin{aligned} W_{ij}q_{j1}|^C &= 0 \\ W_{ijk}q_{j1}q_{k1} + 2W'_{ij}q_{j1}\Lambda_1 + W_{ij}q_{j11}|^C &= 0, \end{aligned}\right\} \tag{11}$$

etc.

Since $q_{11} = 1$ and assuming as before that $q_1$ is a suitable independent variable, we can now solve the first equilibrium equation for the rates $q_{s1}^C$ where $s \neq 1$.

Multiplying the $i$th equation of the second set by $q_{i1}^C$ and adding the $n$ equations we have

$$W_{ijk}q_{i1}q_{j1}q_{k1} + 2W'_{ij}q_{i1}q_{j1}\Lambda_1 + W_{ij}q_{i1}q_{j11}|^C = 0, \tag{12}$$

and observing that the last term vanishes by virtue of the first-order equilibrium equation we find

$$\Lambda_1^C = -\left.\frac{W_{ijk}q_{i1}q_{j1}q_{k1}}{2W'_{ij}q_{i1}q_{j1}}\right|^C. \tag{13}$$

Here we have assumed that the critical equilibrium state is *simple* in the sense that the expression $W'_{ij}q_{i1}q_{j1}|^C$, which clearly plays a key role in the analysis, is non-zero.

We see that this latter manipulation has given us the slope $\Lambda_1^C$ in terms of the rates $q_{j1}^C$, the second derivatives $q_{j11}^C$ having been eliminated. We can thus write the general result:

*The slope of the post-buckling path passing through a simple discrete critical point on a plot of the loading parameter against a (suitable) generalized coordinate is not dependent on the solution of the second-order equilibrium equations.*

A consequence of this result is that a one-degree-of-freedom non-linear Rayleigh–Ritz analysis employing the linear buckling mode will yield the correct value for this slope [28].

Returning to the analysis, knowing $\Lambda_1^C$ and remembering that $q_{111} = 0$, we can now solve the second-order equilibrium equations for the second derivatives $q_{s11}^C$ ($s \neq 1$) if more information is required.

This sequence can now be repeated as many times as we please. Thus multiplying the third-order equilibrium equation by $q_{i1}^C$ and adding we can now find $\Lambda_{11}^C$ without having to solve these equations for $q_{j111}^C$. We can thus write the general result:

*For the post-buckling path passing through a simple discrete critical point the* $m$th *derivative of the loading parameter with respect to a (suitable) generalized coordinate is not dependent on the solution of the* $(m+1)$th-*order equilibrium equations.*

The work so far is applicable whether or not the slope $\Lambda_1^C$ is zero. When $\Lambda_1^C$ is non-zero we have the previously discussed *asymmetric* point of bifurcation [20], and we shall now set $\Lambda_1^C = 0$ to study the *symmetric* points of bifurcation in more detail.

Thus with $\Lambda_1^C = 0$, the third-order equilibrium equation evaluated at the critical point can be written, after multiplying by $q_{i1}^C$ and summing, as

$$W_{ijkl}q_{i1}q_{j1}q_{k1}q_{l1} + 3W_{ijk}q_{i1}q_{j1}q_{k11} + 3W'_{ij}q_{i1}q_{j1}\Lambda_{11} + W_{ij}q_{i1}q_{j111}|^C = 0. \tag{14}$$

As before the last term vanishes by virtue of the first-order equilibrium equation, and with our previous assumption that $W'_{ij}q_{i1}q_{j1}|^C \neq 0$ we can now write the path curvature as

$$\Lambda_{11}^C = -\left.\frac{W_{ijkl}q_{i1}q_{j1}q_{k1}q_{l1} + 3W_{ijk}q_{i1}q_{j1}q_{k11}}{3W'_{ij}q_{i1}q_{j1}}\right|^C. \tag{15}$$

## 3. THEORY FOR THE SPECIALIZED SYSTEM

We shall now consider an important specialization of the preceding general theory which can be made when the original total potential energy function $V(Q_i, \Lambda)$ is linear in the loading parameter $\Lambda$.

When this linearity holds we shall, for purely semantic reasons, replace $\Lambda$ by $P$ and then write

$$V(Q_i, P) = U(Q_i) - P\mathscr{E}(Q_i). \tag{16}$$

Here the function $U(Q_i)$ can be regarded as a *generalized strain energy*, while $P$ can be regarded as the magnitude of a *generalized force* acting through the *generalized displacement* $\mathscr{E}(Q_i)$.

The general theory of Section 2 is of course applicable whether or not this linearity holds, and it remains to examine the post-buckling response of the specialized system on a plot of the load $P$ against its corresponding deflection $\mathscr{E}(Q_i)$.

The energy transformation of the general theory can first be written out in full as follows:

$$\begin{aligned} W(q_i, P) &\equiv V[Q_i^F(P) + q_i, P] \\ &\equiv U[Q_i^F(P) + q_i] - P\mathscr{E}[Q_i^F(P) + q_i], \end{aligned} \tag{17}$$

and some required derivatives of $W$ can be written down,

$$\left.\begin{aligned} W_i &= U_i - P\mathscr{E}_i, \\ W_{ij} &= U_{ij} - P\mathscr{E}_{ij}, \\ W' &= U_i Q_i^{F'} - \mathscr{E} - P\mathscr{E}_i Q_i^{F'}, \\ W'_j &= U_{ij} Q_i^{F'} - \mathscr{E}_j - P\mathscr{E}_{ij} Q_i^{F'}. \end{aligned}\right\} \tag{18}$$

Now $W_j'^C = 0$, so we have

$$\left.\begin{aligned} \mathscr{E}_j^C &= (U_{ij} - P\mathscr{E}_{ij}) Q_i^{F'}|^C \\ &= W_{ij} Q_i^{F'}|^C \end{aligned}\right\} \tag{19}$$

and multiplying by $q_{j1}^C$ and adding we find

$$\mathscr{E}_j q_{j1}|^C = W_{ij} Q_i^{F'} q_{j1}|^C. \tag{20}$$

But the right-hand side of this last equation is zero by virtue of the first-order equilibrium equations, so we have the important result that

$$\mathscr{E}_j q_{j1}|^C = 0. \tag{21}$$

We consider now the change in the corresponding deflection defined by the equation

$$e(q_i, P) \equiv \mathscr{E}[Q_i^F(P) + q_i] - \mathscr{E}[Q_i^F(P)], \tag{22}$$

and we write down some required derivatives as follows:

$$\left.\begin{aligned} e_i = \mathscr{E}_i \qquad e_{ij} &= \mathscr{E}_{ij} \\ e' = \mathscr{E}_i Q_i^{F'} - \mathscr{E}_i Q_i^{F'} \qquad e'^C &= 0 \\ e'' = \mathscr{E}_{ij} Q_i^{F'} Q_j^{F'} + \mathscr{E}_i Q_i^{F''} - \mathscr{E}_{ij} Q_i^{F'} Q_j^{F'} &- \mathscr{E}_i Q_i^{F''} \\ e''^C = 0 \qquad e'_j &= \mathscr{E}_{jk} Q_k^{F'}. \end{aligned}\right\} \tag{23}$$

The total variation of $e$ with $q_1$ along the post-buckling equilibrium path is

$$e(q_1) \equiv e[q_j(q_1), P(q_1)], \tag{24}$$

giving on differentiation

$$\left.\begin{aligned} \frac{\mathrm{d}e}{\mathrm{d}q_1} &= e_j q_{j1} + e' P_1 \\ \frac{\mathrm{d}^2 e}{\mathrm{d}q_1^2} &= (e_{jk} q_{k1} + e'_j P_1) q_{j1} + e_j q_{j11} + (e'_j q_{j1} + e'' P_1) P_1 + e' P_{11}, \end{aligned}\right\} \tag{25}$$

so using the earlier results we have

$$\left.\begin{aligned} \left.\frac{\mathrm{d}e}{\mathrm{d}q_1}\right|^C &= 0 \\ \left.\frac{\mathrm{d}^2 e}{\mathrm{d}q_1^2}\right|^C &= \mathscr{E}_{jk} q_{j1} q_{k1} + 2P_1 \mathscr{E}_{jk} Q_k^{F'} q_{j1} + \mathscr{E}_j q_{j11}|^C. \end{aligned}\right\} \tag{26}$$

For the asymmetric point of bifurcation in which $P_1^C \neq 0$ the total post-buckling variation of $e$ with $P$ is well-behaved and we can write

$$e(q_1) = e[P(q_1)] \tag{27}$$

so that

$$\left.\begin{aligned} \frac{\mathrm{d}e}{\mathrm{d}q_1} &= \frac{\mathrm{d}e}{\mathrm{d}P} P_1 \\ \frac{\mathrm{d}^2 e}{\mathrm{d}q_1^2} &= \frac{\mathrm{d}^2 e}{\mathrm{d}P^2} P_1^2 + \frac{\mathrm{d}e}{\mathrm{d}P} P_{11}. \end{aligned}\right\} \tag{28}$$

We thus obtain the required results

$$\left.\frac{\mathrm{d}e}{\mathrm{d}P}\right|^C = 0, \tag{29}$$

$$\left.\frac{\mathrm{d}^2 e}{\mathrm{d}P^2}\right|^C = \left.\frac{\mathscr{E}_{jk} q_{j1} q_{k1} + 2P_1 \mathscr{E}_{jk} Q_k^{F'} q_{j1} + \mathscr{E}_j q_{j11}}{P_1^2}\right|^C. \tag{30}$$

For the symmetric point of bifurcation in which $P_1^C = 0$ the total post-buckling variation of $e$ with $P$ is associated with a singularity but we can write

$$\frac{\Delta e}{\Delta P} = \frac{\frac{1}{2}(\mathrm{d}^2 e/\mathrm{d}q_1^2) q_1^2 + \dots}{\frac{1}{2} P_{11} q_1^2 + \dots} \tag{31}$$

to obtain the limiting slope

$$\begin{aligned} \left.\frac{\Delta e}{\Delta P}\right|^C &= \left.\frac{\mathrm{d}^2 e/\mathrm{d}q_1^2}{P_{11}}\right|^C \\ &= \left.\frac{\mathscr{E}_{jk} q_{j1} q_{k1} + \mathscr{E}_j q_{j11}}{P_{11}}\right|^C. \end{aligned} \tag{32}$$

## 4. APPLICATIONS

The general theory has been applied to two branching problems, each of which represents an example of the specialized system of Section 3. The first is the rotationally-symmetric branching of a complete spherical shell under uniform external pressure which yields an *asymmetric* point of bifurcation from the trivial fundamental path associated with the uniform contraction of the shell. The second is the branching of a centrally-loaded arch which deforms symmetrically to give a non-linear fundamental equilibrium path from which non-symmetric deformations can develop at an *unstable-symmetric* point of bifurcation.

The spherical shell was analysed first by expanding the deflected form in Legendre functions following the classical buckling analyses. This expansion diagonalizes the basic matrix $W^C_{ij}$ so a direct solution of the successive linear equilibrium equations was possible. The required Legendre integrals were expressed as finite series, so essentially exact expressions for the path derivatives could be obtained by summing these series on a computer. The first three path derivatives were obtained in this way, the first derivative agreeing with that of Thompson [5].

The same problem was then studied using the finite-element Rayleigh–Ritz procedure [29, 30 and 31]. Third-order polynomials were used to construct the form of the lateral deflection which was thus continuous in displacement and first derivative while admitting discontinuities in its second and higher derivatives at the stations. In contrast the in-plane tangential displacement was taken to be linear between the stations at which discontinuities in the first derivative were thus admitted. With this assumed form the basic matrix $W^C_{ij}$ was of course not diagonal, and the general theory was employed, the successive sets of linear equations being solved on a computer. The first three path derivatives were again calculated and were observed to converge rapidly and monotonically to those of the first analysis as the number of stations was increased.

The pinned circular arch under a point load at its centre differs from the spherical shell in having a non-linear fundamental equilibrium path which must be located before the branching analysis can be made. The arch was analysed using the finite-element Rayleigh–Ritz procedure and the fundamental path was determined using a perturbation technique [27] in conjunction with the Newton–Raphson procedure. The vanishing of the stability determinant on the fundamental path before the occurrence of a limit point served to locate the required branching point, and the first- and second-order branching equations were then solved using a digital computer. The specialized theory of Section 3 was then used to obtain the limiting post-buckling slope on a plot of the load against its corresponding deflection, namely the central deflection of the arch.

The analyses of the shell and arch are presented more fully in two companion papers [21, 22].

## 5. DISCUSSION AND CONCLUSIONS

The analytical advantages of a perturbation approach to problems of elastic post-buckling are well known, the initial non-linear problem being reduced to a sequence of non-singular linear problems. Such a branching study has a further intrinsic advantage in that it may pin-point significant local behaviour which can be missed in a more-coarse large-deflection analysis, this point being well illustrated by the work of Hutchinson on the initial post-buckling of oval cylindrical shells under axial compression [16].

Once a perturbation branching approach has been adopted for the analysis of a discrete structural system it is felt, in the light of the two specific analyses, that the present general theory is of great and immediate value since the summations of the theory are readily made on a digital computer.

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(*Received* 3 *May* 1968; *revised* 7 *October* 1968)

**Абстракт**—Представляется общая теория возмущения для анализа разветвления дискретных консервативных систем конструкций. Системы этого рода лучше всего анализировать без помощи схемы диагонализации. Отсуствие такой схемы отличает современные выводы от предидущих исследований. Тензорный запис и система передвигающихся осей, используемые в предидущих исследованиях, имеют также и в настоящее время важное значение, и поэтому они сохраняются. Представляется теория как для общего класса систем, так и для специальных. Предлагаются некоторые виды схем возмущения. Дается окончательно использование общей теории двух задач разветвления и обсуждаются их достоинства.